

T-universal Functions With Prescribed Approximation Curves

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Abstract

Let \mathcal{C} be a family of curves in the unit disc. We show that the set of all functions f holomorphic on the unit disc, which satisfy the following condition, is G_δ and dense in the space of all functions holomorphic on the unit disc.

For each compact set K with connected complement, each function g continuous on K and holomorphic on its interior, every point ζ_0 on the unit circle, every curve $C \in \mathcal{C}$ (ending in ζ_0) and any $\varepsilon > 0$ there exist numbers $0 < a < 1$ and $b \in C$ such that

$$\max_{z \in K} |f(az + b) - g(z)| < \varepsilon \text{ and } |b - \zeta_0| < \varepsilon$$

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1 Introduction

The unit disc will be denoted by $\mathbb{D} = \{z : |z| < 1\}$. The family of compact subsets of \mathbb{C} with connected complement we will denote by \mathcal{M} . For any compact set $K \subset \mathbb{C}$ we will write $A(K)$ for the class of all functions continuous on K and holomorphic in its interior. The space of all functions holomorphic on \mathbb{D} endowed with the usual topology of uniform convergence on compact subsets of \mathbb{D} will be denoted by $H(\mathbb{D})$.

In 1976 Luh[1] proved the existence of a T-universal function Φ on the unit disc. T-universality means that translation in the function's Φ argument forces the function Φ to approximate any function $g \in A(K)$ for $K \in \mathcal{M}$. More precisely, there exist sequences $0 < a_n \rightarrow 0$ and $\{b_n\}_n \subset \mathbb{D}$ such that for any $\zeta \in \partial\mathbb{D}$, any $K \in \mathcal{M}$ and any $g \in A(K)$ there exists a strictly increasing sequence of natural numbers $\{n_k\}_k$

$$\begin{aligned} a_{n_k} z + b_{n_k} &\rightarrow \zeta \quad (k \rightarrow \infty) \text{ for all } z \in K \\ \max_{z \in K} |\Phi(a_{n_k} z + b_{n_k}) - g(z)| &\rightarrow 0 \quad (k \rightarrow \infty). \end{aligned}$$

A similar result has already been proven by Seidel and Walsh[7] in 1941. For further results the reader is referred to [2] and [4].

A question arising in this context is whether the points $\{b_n\}_n$ above can be chosen to lie on any curve C belonging to a prescribed family of curves \mathcal{C} . Tenthoff[9] already gave a positive answer on this question by constructing a function f satisfying the above conditions such that the b_n can be chosen on any radius $\{z = re^{i\varphi}; 0 \leq r < 1\}, \varphi \in [0, 2\pi)$ of the unit disc. We will proof this result for general families of curves

2 T-universal Functions With Prescribed Approximation Curves on the Unit Disc

2.1 Continuous Families of Curves

First we define the notion of a general family of curves. Right after we will restrict our considerations to those families of curves, which we will call continuous ones.

Definition 2.1 Let be $I, J \subset \mathbb{R}$ intervals and $z_\alpha : I \longrightarrow \mathbb{C}$ a continuous function for each $\alpha \in J$ which satisfies the following conditions

$$\lim_{t \rightarrow \inf(I)} z_\alpha(t) = 0 \quad \text{und} \quad \lim_{t \rightarrow \sup(I)} z_\alpha(t) = \infty.$$

Then we will call the family of functions $(z_\alpha)_{\alpha \in J}$ together with the intervals I and J a *general family of curves (from zero to infinity)*. Our short notation will be $\{z_\alpha; I, J\}$.

Definition 2.2 Let be $I, J \subset \mathbb{R}$ intervals and $z_\alpha : I \longrightarrow \mathbb{C}$ a continuous function for each $\alpha \in J$ which satisfies the following conditions

- (1) $\lim_{t \rightarrow \inf(I)} z_\alpha(t) = z_0 \in \mathbb{D} \quad \text{and} \quad \lim_{t \rightarrow \sup(I)} z_\alpha(t) \in \partial\mathbb{D},$
- (2) for each $\zeta \in \partial\mathbb{D}$ there exists an $\alpha \in J$ such that $\lim_{t \rightarrow \sup(I)} z_\alpha(t) = \zeta.$

where $z_0 \in \mathbb{D}$ is fixed and the same for each $\alpha \in J$. Then we will call the family of functions $(z_\alpha)_{\alpha \in J}$ together with the intervals I and J a *general family of curves (in the unit disc \mathbb{D})*. Also here our short notation is $\{z_\alpha; I, J\}$.

Definition 2.3 Let be $I \subset \mathbb{R}$ an interval and $x, y : I \longrightarrow \mathbb{C}$ two continuous bounded functions. We set $C_x = x(I)$ und $C_y = y(I)$ and define the *r-distance between C_x and C_y* as the number

$$\text{r-dist}(C_x, C_y) = \max_{t \in I} |x(t) - y(t)|.$$

Definition 2.4 A general family of curves $\mathcal{C} = \{z_\alpha; I, J\}$ from zero to infinity is to be called *continuous*, if there will exist a finite or countable subset $\tilde{J} \subset J$ such that for all $\delta > 0, \alpha \in J$ and $j \in \mathbb{N}$ there exists an $\tilde{\alpha} \in \tilde{J}$ satisfying the following condition:

$$\text{r-dist}(z_\alpha(I) \cap \{z : |z| \leq j\}, z_{\tilde{\alpha}}(I) \cap \{z : |z| \leq j\}) < \delta.$$

Next we give a very simple sufficient criterium for a family of curves to be continuous.

Theorem 2.5 *Let be $\mathcal{C} = \{z_\alpha; I, J\}$ a family of curves from zero to infinity such that the mapping $(\beta, t) \mapsto z_\beta(t)$ is continuous on $J \times I$. Then \mathcal{C} is a continuous family of curves.*

Proof:

We denote by D the points of ∂J belonging to J and define $\tilde{J} = (J \cap \mathbb{Q}) \cup D$. Then \tilde{J} is countable.

Let be given $\delta > 0, \alpha \in J, j \in \mathbb{N}$. Without loss of generality we may assume that $\delta < 1$.

For any number $M \in \mathbb{N}$ we define depending on I the following interval

$$I_M = \begin{cases} (\inf(I), \inf(I) + \frac{1}{M}) & , \text{ if } \inf(I) \in (-\infty, \infty), \inf(I) \in I^c \\ (-\infty, -M) & , \text{ if } \inf(I) = -\infty \\ [\inf(I), \inf(I) + \frac{1}{M}) & , \text{ if } \inf(I) = \min(I) \end{cases}$$

By the requirement of the theorem we can choose an $M \in \mathbb{N}$ such that

$$\sup\{|z_\beta(t)|; t \in I_M, |\beta - \alpha| \leq 1\} < \frac{\delta}{2}.$$

We will fix this M . Furthermore we set

$$t_\alpha = \sup\{t \in I : |z_\beta(t)| < 2j \text{ for all } \beta \text{ such that } |\beta - \alpha| \leq 1\}.$$

Since $z_\alpha(I)$ is a curve from zero to infinity we have $t_\alpha \in (\inf(I), \sup(I))$. With this number we set $I_\alpha = [\sup(I_M), t_\alpha]$.

Without loss of generality we can assume $\alpha \notin \partial J$ (otherwise we would have $\alpha \in D \subset \tilde{J}$ and were finished). Thus there exists an $\eta \in (0, 1)$ such that $[\alpha - \eta, \alpha + \eta] \subset J$.

Now $(\beta, t) \mapsto z_\beta(t)$ is continuous and hence uniformly continuous on the compact set $[\alpha - \eta, \alpha + \eta] \times I_\alpha$. Thus there exists an $\varepsilon \in (0, 1)$, which satisfies the condition

$$|z_\beta(s) - z_\gamma(t)| < \frac{2\delta}{3}$$

for all $\beta, \gamma \in [\alpha - \eta, \alpha + \eta], s, t \in I_\alpha$ with $|\beta - \gamma| < \varepsilon$ and $|s - t| < \varepsilon$

Particularly we have

$$|z_\alpha(t) - z_\beta(t)| < \frac{2\delta}{3} \text{ for all } \beta \in (\alpha - \varepsilon, \alpha + \varepsilon), t \in I_\alpha.$$

Now we choose an $\tilde{\alpha} \in (\alpha - \varepsilon, \alpha + \varepsilon)$. Then $\tilde{\alpha} \in \tilde{J}$ and we have

$$\max_{t \in I_\alpha} |z_\alpha(t) - z_{\tilde{\alpha}}(t)| \leq \frac{2\delta}{3} < \delta$$

and by the definition of M

$$\max_{t \in I_M} |z_\alpha(t) - z_{\tilde{\alpha}}(t)| \leq \max_{t \in I_M} |z_\alpha(t)| + \max_{t \in I_M} |z_{\tilde{\alpha}}(t)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Furthermore by definition of I_α and I_M the following holds

$$\tilde{I} = z_\alpha^{-1}(\{z : |z| \leq j\}) \cup z_{\tilde{\alpha}}^{-1}(\{z : |z| \leq j\}) \subset I_M \cup I_\alpha.$$

By the estimations above we obtain

$$\max_{t \in \tilde{I}} |z_\alpha(t) - z_{\tilde{\alpha}}(t)| < \delta$$

and this means

$$\text{r-dist}(z_\alpha(I) \cap \{z : |z| \leq j\}, z_{\tilde{\alpha}}(I) \cap \{z : |z| \leq j\}) < \delta,$$

what proves the theorem.

For the sake of transparency and concreteness we give some examples of continuous families of curves on the unit disc:

- (1) The family of all radii $\{z_\alpha(t) = te^{i\alpha}; t \in [0, 1)\}_{\alpha \in [0, 2\pi)}$.
- (2) Logarithmic spirals $\{z_\alpha(t) = e^{(1+i\alpha)t}; t \in (-\infty, \infty)\}_{\alpha \in \mathbb{R}}$, restricted to the unit disc.
- (3) Only one spiral (condition (2) in definition 2.2 can be weakened):
 $z(t) = (1 - e^{-t})e^{it}, t > 0.$

2.2 Main Result

First, let be $\mathcal{C} = \{z_\alpha; I, J\}$ a fixed continuous family of curves in the unit disc. Now we are going to define a class of functions we will prove to be nonempty and moreover a dense G_δ -set in the space $H(\mathbb{D})$.

The set of all functions $f \in H(\mathbb{D})$ such that for every $K \in \mathcal{M}$, every function $g \in A(K)$, each $\varepsilon > 0$, each $\zeta_0 \in \partial\mathbb{D}$ and any curve $C \in \mathcal{C}$ (ending in ζ_0) there exist numbers $0 < a < 1$ and $b \in C$ such that

$$\max_{z \in K} |f(az + b) - g(z)| < \varepsilon, \quad |b - \zeta_0| < \varepsilon$$

will be called the *class of T -universal functions with respect to the family of curves \mathcal{C} in \mathbb{D}* . It is denoted by $\mathcal{U}_{\mathcal{C}}(\mathbb{D})$.

Note that the definition above already implies $aK + b \subset \mathbb{D}$, otherwise the function f would not be defined on $aK + b$.

Theorem 2.6 *The set $U_{\mathcal{C}}(\mathbb{D})$ of T -universal functions on the unit disc with prescribed approximation curves is G_{δ} and dense in $H(\mathbb{D})$.*

2.2.1 Proof of Theorem 2.6

We will fix some sequences and sets for abbreviation purposes.

- (1) For each $m \in \mathbb{N}$ we denote $L_m = \{z : |z| \leq m\}$.
- (2) Let be $\{p_j\}_j$ an enumeration of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$.
- (3) The sequence $\{\zeta_p\}_p$ is chosen to be dense on the unit circle $\partial\mathbb{D}$.
- (4) For each $p \in \mathbb{N}$ we choose with respect to those curves ending in ζ_p a sequence of curves $\{C_{pl}\}_l$ according to definition 2.4. I.e. for each curve $C \in \mathcal{C}$ ending in ζ_p we find an index $l \in \mathbb{N}$ such that C_{pl} lies arbitrarily near to C in terms of the r-distance.
- (5) For $p, l \in \mathbb{N}$ we choose a sequence of points $\{b_{nlp}\}_n$ being dense on C_{pl} .
- (6) The sequence $\{a_k\}_k$ is a sequence of positive numbers dense in $(0, 1)$.

With this notions we will prove three technical lemmas. For an intermediate step we need an auxiliary class. For this we fix an $h \in \mathbb{N}$. The set of all functions $f \in H(\mathbb{D})$ such that for every $K \in \mathcal{M}(\mathbb{C})$, every function $g \in A(K)$, each $\varepsilon > 0$, each $\zeta_0 \in \partial\mathbb{D}$ and any curve $C \in \mathcal{C}$ (ending in ζ_0) there exist numbers $0 < a < 1$ and $b \in U_{\frac{1}{h}}(C) = \{z \in \mathbb{C} : \text{dist}(z, C) < \frac{1}{h}\}$ such that

$$\max_{z \in K} |f(az + b) - g(z)| < \varepsilon, \quad |b - \zeta_0| < \varepsilon$$

will be denoted by $\mathcal{U}_{\mathcal{C}}^{(h)}(\mathbb{D})$.

For $m, j, p, s, t, l, k, n \in \mathbb{N}$ we set

$$\mathcal{O}_{\mathcal{C}}(m, j, p, s, t, l, k, n) = \left\{ g \in H(\mathbb{D}) : \max_{z \in L_m} |g(a_k z + b_{nlp}) - p_j(z)| < \frac{1}{s}; \right. \\ \left. b_{nlp} \in C_{lp}, |b_{nlp} - \zeta_p| < \frac{1}{t} \right\}$$

Note that this set depends on \mathcal{C} , although it does not appear itself in the above definition. But the C_{pl} are chosen in \mathcal{C} .

Our first lemma states that the class $\mathcal{U}_{\mathcal{C}}(\mathbb{D})$ has a representation with intersections and unions of the sets $\mathcal{O}_{\mathcal{C}}(m, j, p, s, t, l, k, n)$.

Lemma 2.7 *The following equations hold*

$$\mathcal{U}_{\mathcal{C}}(\mathbb{D}) = \bigcap_{h=1}^{\infty} \mathcal{U}_{\mathcal{C}}^{(h)}(\mathbb{D}) = \bigcap_{m,j,p,s,t,l=1}^{\infty} \bigcup_{k,n=1}^{\infty} \mathcal{O}_{\mathcal{C}}(m, j, p, s, t, l, k, n).$$

Proof:

The first equation is obvious due to the definition of the considered classes.

Let be f an element of the right hand side and let be given $K \in \mathcal{M}, g \in A(K), \varepsilon > 0, \zeta_0 \in \partial\mathbb{D}$ and a curve $C \in \mathcal{C}$ ending in ζ_0 . We fix an $h \in \mathbb{N}$.

Then there is an $m \in \mathbb{N}$ such that $K \subset L_m$. Furthermore we find $s, t \in \mathbb{N}$ such that $\frac{1}{s} < \frac{\varepsilon}{2}, \frac{1}{t} < \frac{\varepsilon}{2}$. Then by Mergelyan's theorem we choose a $j \in \mathbb{N}$ satisfying

$$\max_K |p_j(z) - g(z)| < \frac{\varepsilon}{2}.$$

Since the sequence $\{\zeta_p\}_s$ is dense in $\partial\mathbb{D}$ there is a $p \in \mathbb{N}$ with $|\zeta_p - \zeta_0| < \frac{\varepsilon}{2}$.

The family of curves \mathcal{C} is continuous, so we can find an $l \in \mathbb{N}$ to this p satisfying

$$\text{r-dist}(C, C_{pl}) < \frac{1}{h}.$$

Due to the definition of $\mathcal{O}_{\mathcal{C}}(m, j, p, s, t, l, k, n)$ and the representation of the right hand side there exist numbers $n, k \in \mathbb{N}$ with the following properties

$$\max_{L_m} |f(a_k z + b_{nlp}) - p_j(z)| < \frac{1}{s} \quad \text{and} \quad b_{nlp} \in C_{pl}, |b_{nlp} - \zeta_p| < \frac{1}{t}.$$

Hence we obtain

$$\max_K |f(a_k z + b_{nlp}) - g(z)| \leq \\ \max_{L_m} |f(a_k z + b_{nlp}) - p_j(z)| + \max_K |p_j(z) - g(z)| < \frac{1}{s} + \frac{\varepsilon}{2} < \varepsilon$$

Since $\text{r-dist}(C, C_{pl}) < \frac{1}{h}$ and $b_{nlp} \in C_{pl}$ it is also true that $b_{nlp} \in U_{\frac{1}{h}}(C)$ and hence $f \in \mathcal{U}_C^{(h)}(\mathbb{D})$. Since $h \in \mathbb{N}$ was arbitrary we conclude $f \in \mathcal{U}_C(\mathbb{D})$.

Now let be f a function lying in $\mathcal{U}_C(\mathbb{D})$ and let be given

$m, j, p, s, t, l \in \mathbb{N}$. By definition of $\mathcal{U}_C(\mathbb{D})$ there exist $0 < a < 1$ and $b \in C_{pl}$ satisfying

$$\max_{L_m} |f(az + b) - p_j(z)| < \frac{1}{2s} \quad \text{und} \quad |b - \zeta_p| < \frac{1}{2t}.$$

If we set $d = \text{dist}(aL_m + b, \partial\mathbb{D}) > 0$ and

$$\tilde{L}_m = \left\{ z \in \mathbb{C} : \text{dist}(az + b, \partial\mathbb{D}) \geq \frac{d}{2}, az + b \in \mathbb{D} \right\},$$

then $a\tilde{L}_m + b$ will be a compact subset of \mathbb{D} with $a\tilde{L}_m + b \supset aL_m + b$. Since f is uniformly continuous on this compact set, there exists a $\delta > 0$ such that $|f(z_1) - f(z_2)| < \frac{1}{2s}$ for all $z_1, z_2 \in a\tilde{L}_m + b$, $|z_1 - z_2| < \delta$.

Then we find numbers $k, n \in \mathbb{N}$ with $|a_k - a| < \frac{\delta}{2m}$ and $|b_{nlp} - b| < \min\left\{\frac{\delta}{2}, \frac{1}{2t}\right\}$. Thus for all $z \in L_m$ we have $|a_k z + b_{nlp} - (az + b)| < \delta$ and hence we obtain

$$\begin{aligned} \max_{L_m} |f(a_k z + b_{nlp}) - p_j(z)| &\leq \\ \max_{L_m} |f(a_k z + b_{nlp}) - f(az + b)| + \max_{L_m} |f(az + b) - p_j(z)| &< \frac{1}{2s} + \frac{1}{2s} = \frac{1}{s} \end{aligned}$$

Finally we have $|b_{nlp} - \zeta_p| \leq |b_{nlp} - b| + |b - \zeta_p| < \frac{1}{2t} + \frac{1}{2t} = \frac{1}{t}$ and hence f lies in the right hand side of the stated equation. This proves the lemma.

The next lemma states, taken together with the preceding one, that $\mathcal{U}_C(\mathbb{D})$ is indeed a G_δ set in $H(\mathbb{D})$

Lemma 2.8 *For all $m, j, p, s, t, l, k, n \in \mathbb{N}$ the set $\mathcal{O}_C(m, j, p, s, t, l, k, n)$ is open in $H(\mathbb{D})$.*

Proof:

Fix $m, j, p, s, t, l, k, n \in \mathbb{N}$ and $f \in \mathcal{O}_C(m, j, p, s, t, l, k, n)$. We set

$$\delta = \frac{1}{s} - \max_{L_m} |f(a_k z + b_{nlp}) - p_j(z)| > 0$$

and define

$$U_\delta(f) = \left\{ g \in H(\mathbb{D}) : \max_{L_m} |g(a_k z + b_{nlp}) - f(a_k z + b_{nlp})| < \delta \right\}.$$

Then we obtain for all $g \in U_\delta(f)$:

$$\begin{aligned} \max_{L_m} |g(a_k z + b_{nlp}) - p_j(z)| &\leq \\ \max_{L_m} |g(a_k z + b_{nlp}) - f(a_k z + b_{nlp})| + \max_{L_m} |f(a_k z + b_{nlp}) - p_j(z)| &< \\ \frac{1}{s} - \max_{L_m} |f(a_k z + b_{nlp}) - p_j(z)| + \max_{L_m} |f(a_k z + b_{nlp}) - p_j(z)| &= \frac{1}{s} \end{aligned}$$

Thus the open δ -neighborhood $U_\delta(f)$ of f is contained in $\mathcal{O}_C(m, j, p, s, t, l, k, n)$ and the statement follows.

Lemma 2.9 *For all $m, j, p, s, t, l \in \mathbb{N}$ the set*

$$\bigcup_{k, n=1}^{\infty} \mathcal{O}_C(m, j, p, s, t, l, k, n)$$

is dense in the space $H(\mathbb{D})$.

Proof:

Fix numbers $m, j, p, s, t, l \in \mathbb{N}$ and let be given $f \in H(\mathbb{D})$, a compact set $K \subset \mathbb{D}$ and an $\varepsilon > 0$.

First we find a compact set $K \subset B \subset \mathbb{D}$ with connected complement in \mathbb{D} . Since B is a compact subset of \mathbb{D} we can choose a $\delta > 0$ such that

$\{z : |z - \zeta_p| < \delta\} \cap B = \emptyset$. Since C_{pl} is a curve ending on the unit circle and $\{a_k\}_k$ is dense in the interval $(0, 1)$ we can choose numbers $k, n \in \mathbb{N}$ such that $|b_{nlp} - \zeta_p| < \min\{\frac{1}{t}, \frac{\delta}{2}\}$ and $0 < a_k < \frac{\delta}{2m}$. By definition the point b_{nlp} lies in C_{pl} anyway.

Thus we have for all $z \in L_m$:

$$|a_k z + b_{nlp} - \zeta_p| \leq a_k |z| + |b_{nlp} - \zeta_p| < \frac{\delta}{2m} m + \frac{\delta}{2} = \delta$$

and hence

$$a_k L_m + b_{nlp} \subset \{z : |z - \zeta_p| < \delta\} \subset B^c.$$

Due to Runge's theorem on polynomial approximation there exists a polynomial p with

$$\begin{aligned} \max_B |p(z) - f(z)| &< \varepsilon \\ \max_{L_m} |p(a_k z + b_{nlp}) - p_j(z)| &< \frac{1}{s} \end{aligned}$$

Thus p lies "near to" f and we have $p \in \mathcal{O}_C(m, j, p, s, t, l, k, n)$, what completes the proof.

Now theorem 2.6 is a consequence of the preceding three lemmas. Indeed, lemma 2.7 and 2.8 state that $\mathcal{U}_C(\mathbb{D})$ is a G_δ set in $H(\mathbb{D})$. Together with lemma 2.9 we obtain that $\mathcal{U}_C(\mathbb{D})$ has an representation as a countable intersection of dense sets. Recalling that $H(\mathbb{D})$ is a complete metric space and applying Baire's category theorem we obtain that $\mathcal{U}_C(\mathbb{D})$ is dense in $H(\mathbb{D})$. This proves theorem 2.6.

2.2.2 An additional property of T-universal functions with prescribed approximation curves

Remark 2.10 *Every function $f \in H(\mathbb{D})$ can be expressed as the sum of two T-universal functions with prescribed approximation curves.*

The following short **proof** is due to J.-P. Kahane [3].

Let be given a function $f \in H(\mathbb{D})$. The mapping

$$T_f(g) : H(\mathbb{D}) \rightarrow H(\mathbb{D}), T_f(g) = g + f \quad (g \in H(\mathbb{D}))$$

is a homeomorphism.

Since the set $\mathcal{U}_C(\mathbb{D})$ is G_δ and dense in $H(\mathbb{D})$, the same holds for

$$T_f(\mathcal{U}_C(\mathbb{D})) = \mathcal{U}_C(\mathbb{D}) + f.$$

Due to Baire's category theorem we have

$$\mathcal{U}_C(\mathbb{D}) \cap (\mathcal{U}_C(\mathbb{D}) + f) \neq \emptyset.$$

Thus there exist $g, h \in \mathcal{U}_C(\mathbb{D})$ with $f = g - h$. Since $-h \in \mathcal{U}_C(\mathbb{D})$ the result follows.

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